Example 11.23

Solve the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t} \quad (t \geqslant 0)$$

subject to the initial conditions $x = 1$ and $dx/dt = 0$ at $t = 0$.

Solution

Taking Laplace transforms

$$\mathscr{L}\left\{\frac{d^2x}{dt^2}\right\} + 5\mathscr{L}\left\{\frac{dx}{dt}\right\} + 6\mathscr{L}\{x\} = 2\mathscr{L}\{e^{-t}\}$$

leads to the transformed equation

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 5[sX(s) - x(0)] + 6X(s) = \frac{2}{s+1}$$

which on rearrangement gives

$$(s^2 + 5s + 6)X(s) = \frac{2}{s+1} + (s+5)x(0) + \dot{x}(0)$$

Incorporating the given initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ leads to

$$(s^2 + 5s + 6)X(s) = \frac{2}{s+1} + s + 5$$

That is,

$$X(s) = \frac{2}{(s+1)(s+2)(s+3)} + \frac{s+5}{(s+3)(s+2)}$$

Resolving the rational terms into partial fractions gives

$$X(s) = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3} + \frac{3}{s+2} - \frac{2}{s+3}$$

$$= \frac{1}{s+1} + \frac{1}{s+2} - \frac{1}{s+3}$$

Taking inverse transforms gives the desired solution

$$x(t) = e^{-t} + e^{-2t} - e^{-3t} \quad (t \geqslant 0)$$

In principle the procedure adopted in Example 11.23 for solving a second-order linear differential equation with constant coefficients is readily carried over to higher-order differential equations. A general nth-order linear differential equation may be written as

$$a_n\frac{d^nx}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_0x = u(t) \quad (t \geqslant 0) \qquad \textbf{(11.18)}$$

where $a_n, a_{n-1}, \dots, a_0$ are constants, with $a_n \neq 0$. This may be written in the more concise form

$$q(\mathrm{D})x(t) = u(t) \tag{11.19}$$

where D denotes the operator $\mathrm{d}/\mathrm{d}t$ and $q(\mathrm{D})$ is the polynomial

$$q(\mathrm{D}) = \sum_{r=0}^{n} a_r \mathrm{D}^r$$

The objective is then to determine the response $x(t)$ for a given forcing function $u(t)$ subject to the given set of initial conditions

$$\mathrm{D}^r x(0) = \left[\frac{\mathrm{d}^r x}{\mathrm{d}t^r}\right]_{t=0} = c_r \quad (r = 0, 1, \dots, n-1)$$

Taking Laplace transforms in (11.19) and proceeding as before leads to

$$X(s) = \frac{p(s)}{q(s)}$$

where

$$p(s) = U(s) + \sum_{r=0}^{n-1} c_r \sum_{i=r+1}^{n} a_i s^{i-r-1}$$

Then, in principle, by taking the inverse transform, the desired response $x(t)$ may be obtained as

$$x(t) = \mathscr{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\}$$

For high-order differential equations the process of performing this inversion may prove to be rather tedious, and matrix methods may be used as indicated in Chapter 6 of the companion text, *Advanced Modern Engineering Mathematics*.

To conclude this section, further worked examples are developed in order to help consolidate understanding of this method for solving linear differential equations.

Example 11.24 Solve the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = \sin t \quad (t \geqslant 0)$$

subject to the initial conditions $x = 0$ and $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$.

Solution Taking the Laplace transforms

$$\mathscr{L}\left\{\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right\} + 6\mathscr{L}\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} + 9\mathscr{L}\{x\} = \mathscr{L}\{\sin t\}$$

leads to the equation

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 6[sX(s) - x(0)] + 9X(s) = \frac{1}{s^2 + 1}$$

which on rearrangement gives

$$(s^2 + 6s + 9)X(s) = \frac{1}{s^2 + 1} + (s + 6)x(0) + \dot{x}(0)$$

Incorporating the given initial conditions $x(0) = \dot{x}(0) = 0$ leads to

$$X(s) = \frac{1}{(s^2 + 1)(s + 3)^2}$$

Resolving into partial fractions gives

$$X(s) = \tfrac{3}{50}\frac{1}{s + 3} + \tfrac{1}{10}\frac{1}{(s + 3)^2} + \tfrac{2}{25}\frac{1}{s^2 + 1} - \tfrac{3}{50}\frac{s}{s^2 + 1}$$

that is,

$$X(s) = \tfrac{3}{50}\frac{1}{s + 3} + \tfrac{1}{10}\left[\frac{1}{s^2}\right]_{s \to s+3} + \tfrac{2}{25}\frac{1}{s^2 + 1} - \tfrac{3}{50}\frac{s}{s^2 + 1}$$

Taking inverse transforms, using the shift theorem, leads to the desired solution

$$x(t) = \tfrac{3}{50}e^{-3t} + \tfrac{1}{10}te^{-3t} + \tfrac{2}{25}\sin t - \tfrac{3}{50}\cos t \quad (t \geq 0)$$

Example 11.25 Solve the differential equation

$$\frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 17\frac{dx}{dt} + 13x = 1 \quad (t \geq 0)$$

subject to the initial conditions $x = dx/dt = 1$ and $d^2x/dt^2 = 0$ at $t = 0$.

Solution Taking Laplace transforms

$$\mathscr{L}\left\{\frac{d^3x}{dt^3}\right\} + 5\mathscr{L}\left\{\frac{d^2x}{dt^2}\right\} + 17\mathscr{L}\left\{\frac{dx}{dt}\right\} + 13\mathscr{L}\{x\} = \mathscr{L}\{1\}$$

leads to the equation

$$s^3X(s) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0) + 5[s^2X(s) - sx(0) - \dot{x}(0)]$$

$$+ 17[sX(s) - x(0)] + 13X(s) = \frac{1}{s}$$

which on rearrangement gives

$$(s^3 + 5s^2 + 17s + 13)X(s) = \frac{1}{s} + (s^2 + 5s + 17)x(0) + (s + 5)\dot{x}(0) + \ddot{x}(0)$$

Incorporating the given initial conditions $x(0) = \dot{x}(0) = 1$ and $\ddot{x}(0) = 0$ leads to

$$X(s) = \frac{s^3 + 6s^2 + 22s + 1}{s(s^3 + 5s^2 + 17s + 13)}$$

Clearly $s + 1$ is a factor of $s^3 + 5s^2 + 17s + 13$, and by algebraic division we have

$$X(s) = \frac{s^3 + 6s^2 + 22s + 1}{s(s + 1)(s^2 + 4s + 13)}$$

Resolving into partial fractions,

$$X(s) = \frac{\frac{1}{13}}{s} + \frac{\frac{8}{5}}{s+1} - \frac{1}{65}\frac{44s+7}{s^2+4s+13}$$

$$= \frac{\frac{1}{13}}{s} + \frac{\frac{8}{5}}{s+1} - \frac{1}{65}\frac{44(s+2)-27(3)}{(s+2)^2+3^2}$$

Taking inverse transforms, using the shift theorem, leads to the solution

$$x(t) = \tfrac{1}{13} + \tfrac{8}{5}e^{-t} - \tfrac{1}{65}e^{-2t}(44\cos 3t - 27\sin 3t) \quad (t \geqslant 0)$$

11.3.4 Exercise

5 Using Laplace transform methods, solve for $t \geqslant 0$ the following differential equations, subject to the specified initial conditions. (Readers are encouraged to check their solutions using an appropriate software package.)

(a) $\dfrac{dx}{dt} + 3x = e^{-2t}$ subject to $x = 2$ at $t = 0$

(b) $3\dfrac{dx}{dt} - 4x = \sin 2t$ subject to $x = \frac{1}{3}$ at $t = 0$

(c) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 1$

subject to $x = 0$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(d) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 4\cos 2t$

subject to $y = 0$ and $\dfrac{dy}{dt} = 2$ at $t = 0$

(e) $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 2e^{-4t}$

subject to $x = 0$ and $\dfrac{dx}{dt} = 1$ at $t = 0$

(f) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 5x = 3e^{-2t}$

subject to $x = 4$ and $\dfrac{dx}{dt} = -7$ at $t = 0$

(g) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - 2x = 5e^{-t}\sin t$

subject to $x = 1$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(h) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = 3t$

subject to $y = 0$ and $\dfrac{dy}{dt} = 1$ at $t = 0$

(i) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 4x = t^2 + e^{-2t}$

subject to $x = \frac{1}{2}$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(j) $9\dfrac{d^2x}{dt^2} + 12\dfrac{dx}{dt} + 5x = 1$

subject to $x = 0$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(k) $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 16\sin 4t$

subject to $x = -\frac{1}{2}$ and $\dfrac{dx}{dt} = 1$ at $t = 0$

(l) $9\dfrac{d^2y}{dt^2} + 12\dfrac{dy}{dt} + 4y = e^{-t}$

subject to $y = 1$ and $\dfrac{dy}{dt} = 1$ at $t = 0$

(m) $\dfrac{d^3x}{dt^3} - 2\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + 2x = 2 + t$

subject to $x = 0$, $\dfrac{dx}{dt} = 1$ and $\dfrac{d^2x}{dt^2} = 0$ at $t = 0$

(n) $\dfrac{d^3x}{dt^3} + \dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x = \cos 3t$

subject to $x = 0$, $\dfrac{dx}{dt} = 1$ and $\dfrac{d^2x}{dt^2} = 1$ at $t = 0$

11.3.5 Simultaneous differential equations

In engineering we frequently encounter systems whose characteristics are modelled by a set of simultaneous linear differential equations with constant coefficients. The method of solution is essentially the same as that adopted in Section 11.3.3 for solving a single differential equation in one unknown. Taking Laplace transforms throughout, the system of simultaneous differential equations is transformed into a system of simultaneous algebraic equations, which are then solved for the transformed variables; inverse transforms then give the desired solutions.

Example 11.26 Solve for $t \geqslant 0$ the simultaneous first-order differential equations

$$\frac{dx}{dt} + \frac{dy}{dt} + 5x + 3y = e^{-t} \qquad (11.20)$$

$$2\frac{dx}{dt} + \frac{dy}{dt} + x + y = 3 \qquad (11.21)$$

subject to the initial conditions $x = 2$ and $y = 1$ at $t = 0$.

Solution Taking Laplace transforms in (11.20) and (11.21) gives

$$sX(s) - x(0) + sY(s) - y(0) + 5X(s) + 3Y(s) = \frac{1}{s+1}$$

$$2[sX(s) - x(0)] + sY(s) - y(0) + X(s) + Y(s) = \frac{3}{s}$$

Rearranging and incorporating the given initial conditions $x(0) = 2$ and $y(0) = 1$ leads to

$$(s+5)X(s) + (s+3)Y(s) = 3 + \frac{1}{s+1} = \frac{3s+4}{s+1} \qquad (11.22)$$

$$(2s+1)X(s) + (s+1)Y(s) = 5 + \frac{3}{s} = \frac{5s+3}{s} \qquad (11.23)$$

Hence, by taking Laplace transforms, the pair of simultaneous differential equations (11.20) and (11.21) in $x(t)$ and $y(t)$ has been transformed into a pair of simultaneous algebraic equations (11.22) and (11.23) in the transformed variables $X(s)$ and $Y(s)$. These algebraic equations may now be solved simultaneously for $X(s)$ and $Y(s)$ using standard algebraic techniques.

Solving first for $X(s)$ gives

$$X(s) = \frac{2s^2 + 14s + 9}{s(s+2)(s-1)}$$

Resolving into partial fractions,

$$X(s) = -\frac{\frac{9}{2}}{s} - \frac{\frac{11}{6}}{s+2} + \frac{\frac{25}{3}}{s-1}$$

which on inversion gives

$$x(t) = -\tfrac{9}{2} - \tfrac{11}{6}e^{-2t} + \tfrac{25}{3}e^{t} \quad (t \geqslant 0) \qquad (11.24)$$

Likewise, solving for $Y(s)$ gives

$$Y(s) = \frac{s^3 - 22s^2 - 39s - 15}{s(s+1)(s+2)(s-1)}$$

Resolving into partial fractions,

$$Y(s) = \frac{\frac{15}{2}}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{11}{2}}{s+2} - \frac{\frac{25}{2}}{s-1}$$

which on inversion gives

$$y(t) = \tfrac{15}{2} + \tfrac{1}{2}e^{-t} + \tfrac{11}{2}e^{-2t} - \tfrac{25}{2}e^{t} \quad (t \geqslant 0)$$

Thus the solution to the given pair of simultaneous differential equations is

$$\left.\begin{aligned} x(t) &= -\tfrac{9}{2} - \tfrac{11}{6}e^{-2t} + \tfrac{25}{3}e^{t} \\ y(t) &= \tfrac{15}{2} + \tfrac{1}{2}e^{-t} + \tfrac{11}{2}e^{-2t} - \tfrac{25}{2}e^{t} \end{aligned}\right\} \quad (t \geqslant 0)$$

Note: When solving a pair of first-order simultaneous differential equations such as (11.20) and (11.21), an alternative approach to obtaining the value of $y(t)$ having obtained $x(t)$ is to use (11.20) and (11.21) directly.

Eliminating dy/dt from (11.20) and (11.21) gives

$$2y = \frac{dx}{dt} - 4x - 3 + e^{-t}$$

Substituting the solution obtained in (11.24) for $x(t)$ gives

$$2y = (\tfrac{11}{3}e^{-2t} + \tfrac{25}{3}e^{t}) - 4(-\tfrac{9}{2} - \tfrac{11}{6}e^{-2t} + \tfrac{25}{3})e^{t} - 3 + e^{-t}$$

leading as before to the solution

$$y = \tfrac{15}{2} + \tfrac{1}{2}e^{-t} + \tfrac{11}{2}e^{-2t} - \tfrac{25}{2}e^{t}$$

A further alternative is to express (11.22) and (11.23) in matrix form and solve for $X(s)$ and $Y(s)$ using Gaussian elimination.

In principle, the same procedure as used in Example 11.26 can be employed to solve a pair of higher-order simultaneous differential equations or a larger system of differential equations involving more unknowns. However, the algebra involved can become quite complicated, and matrix methods are usually preferred.

11.3.6 Exercise

6 Using Laplace transform methods, solve for $t \geqslant 0$ the following simultaneous differential equations subject to the given initial conditions. (Readers are encouraged to check their solutions using an appropriate software package.)

(a) $$2\frac{dx}{dt} - 2\frac{dy}{dt} - 9y = e^{-2t}$$

$$2\frac{dx}{dt} + 4\frac{dy}{dt} + 4x - 37y = 0$$

subject to $x = 0$ and $y = \frac{1}{4}$ at $t = 0$

(b) $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} + x - y = 5\sin t$

$2\dfrac{dx}{dt} + 3\dfrac{dy}{dt} + x - y = e^t$

subject to $x = 0$ and $y = 0$ at $t = 0$

(c) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = e^{-3t}$

$\dfrac{dy}{dt} + 5x + 3y = 5e^{-2t}$

subject to $x = -1$ and $y = 4$ at $t = 0$

(d) $3\dfrac{dx}{dt} + 3\dfrac{dy}{dt} - 2x = e^t$

$\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - y = 1$

subject to $x = 1$ and $y = 1$ at $t = 0$

(e) $3\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x = 3\sin t + 5\cos t$

$2\dfrac{dx}{dt} + \dfrac{dy}{dt} + y = \sin t + \cos t$

subject to $x = 0$ and $y = -1$ at $t = 0$

(f) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + y = t$

$\dfrac{dx}{dt} + 4\dfrac{dy}{dt} + x = 1$

subject to $x = 1$ and $y = 0$ at $t = 0$

(g) $2\dfrac{dx}{dt} + 3\dfrac{dy}{dt} + 7x = 14t + 7$

$5\dfrac{dx}{dt} - 3\dfrac{dy}{dt} + 4x + 6y = 14t - 14$

subject to $x = y = 0$ at $t = 0$

(h) $\dfrac{d^2x}{dt^2} = y - 2x$

$\dfrac{d^2y}{dt^2} = x - 2y$

subject to $x = 4$, $y = 2$, $dx/dt = 0$ and $dy/dt = 0$ at $t = 0$

(i) $5\dfrac{d^2x}{dt^2} + 12\dfrac{d^2y}{dt^2} + 6x = 0$

$5\dfrac{d^2x}{dt^2} + 16\dfrac{d^2y}{dt^2} + 6y = 0$

subject to $x = \frac{7}{4}$, $y = 1$, $dx/dt = 0$ and $dy/dt = 0$ at $t = 0$

(j) $2\dfrac{d^2x}{dt^2} - \dfrac{d^2y}{dt^2} - \dfrac{dx}{dt} - \dfrac{dy}{dt} = 3y - 9x$

$2\dfrac{d^2x}{dt^2} - \dfrac{d^2y}{dt^2} + \dfrac{dx}{dt} + \dfrac{dy}{dt} = 5y - 7x$

subject to $x = dx/dt = 1$ and $y = dy/dt = 0$ at $t = 0$

(c) $\dfrac{3s-2}{s^2}+\dfrac{4s}{s^2+4}$, Re$(s)>0$

(d) $\dfrac{s}{s^2-9}$, Re$(s)>3$

(e) $\dfrac{2}{s^2-4}$, Re$(s)>2$

(f) $\dfrac{5}{s+2}+\dfrac{3}{s}-\dfrac{2s}{s^2+4}$, Re$(s)>0$

(g) $\dfrac{4}{(s+2)^2}$, Re$(s)>-2$

(h) $\dfrac{4}{s^2+6s+13}$, Re$(s)>-3$

(i) $\dfrac{2}{(s+4)^3}$, Re$(s)>-4$

(j) $\dfrac{36-6s+4s^2-2s^3}{s^4}$, Re$(s)>0$

(k) $\dfrac{2s+15}{s^2+9}$, Re$(s)>0$

(l) $\dfrac{s^2-4}{(s^2+4)^2}$, Re$(s)>0$

(m) $\dfrac{18s^2-54}{(s^2+9)^3}$, Re$(s)>0$

(n) $\dfrac{2}{s^3}-\dfrac{3s}{s^2+16}$, Re$(s)>0$

(o) $\dfrac{2}{(s+2)^3}+\dfrac{s+1}{s^2+2s+5}+\dfrac{3}{s}$, Re$(s)>0$

4 (a) $\frac{1}{4}(e^{-3t}-e^{-7t})$ (b) $-e^{-t}+2e^{3t}$
(c) $\frac{4}{9}-\frac{1}{3}t-\frac{4}{9}e^{-3t}$ (d) $2\cos 2t+3\sin 2t$
(e) $\frac{1}{64}(4t-\sin 4t)$ (f) $e^{-2t}(\cos t+6\sin t)$
(g) $\frac{1}{8}(1-e^{-2t}\cos 2t+3e^{-2t}\sin 2t)$ (h) $e^{t}-e^{-t}-2te^{-t}$
(i) $e^{-t}(\cos 2t+3\sin 2t)$ (j) $\frac{1}{2}e^{t}-3e^{2t}+\frac{11}{2}e^{3t}$
(k) $-2e^{-3t}+2\cos(\sqrt{2}t)-\sqrt{\tfrac{1}{2}}\sin(\sqrt{2}t)$
(l) $\frac{1}{5}e^{t}-\frac{1}{5}e^{-t}(\cos t-3\sin t)$
(m) $e^{-t}(\cos 2t-\sin 2t)$ (n) $\frac{1}{2}e^{2t}-2e^{3t}+\frac{3}{2}e^{-4t}$
(o) $-e^{t}+\frac{3}{2}e^{2t}-\frac{1}{2}e^{-2t}$ (p) $4-\frac{9}{2}\cos t+\frac{1}{2}\cos 3t$
(q) $9e^{-2t}-e^{-3t/2}[7\cos(\frac{1}{2}\sqrt{3}t)-\sqrt{3}\sin(\frac{1}{2}\sqrt{3}t)]$
(r) $\frac{1}{9}e^{-t}-\frac{1}{10}e^{-2t}-\frac{1}{90}e^{-t}(\cos 3t+3\sin 3t)$

5 (a) $x(t)=e^{-2t}+e^{-3t}$
(b) $x(t)=\frac{35}{78}e^{4t/3}-\frac{3}{26}(\cos 2t+\frac{2}{3}\sin 2t)$
(c) $x(t)=\frac{1}{5}(1-e^{-t}\cos 2t-\frac{1}{2}e^{-t}\sin 2t)$
(d) $y(t)=\frac{1}{25}(12e^{-t}+30te^{-t}-12\cos 2t+16\sin 2t)$
(e) $x(t)=-\frac{7}{5}e^{t}+\frac{4}{3}e^{2t}+\frac{1}{15}e^{-4t}$
(f) $x(t)=e^{-2t}(\cos t+\sin t+3)$
(g) $x(t)=\frac{13}{12}e^{t}-\frac{1}{3}e^{-2t}+\frac{1}{4}e^{-t}(\cos 2t-3\sin 2t)$
(h) $y(t)=-\frac{2}{3}+t+\frac{2}{3}e^{-t}[\cos(\sqrt{2}t)+\sqrt{\tfrac{1}{2}}\sin(\sqrt{2}t)]$
(i) $x(t)=(\frac{1}{8}+\frac{3}{4}t)e^{-2t}+\frac{1}{2}t^2e^{-2t}+\frac{3}{8}-\frac{1}{2}t+\frac{1}{4}t^2$
(j) $x(t)=\frac{1}{5}-\frac{1}{5}e^{-2t/3}(\cos\frac{1}{3}t+2\sin\frac{1}{3}t)$
(k) $x(t)=te^{-4t}-\frac{1}{2}\cos 4t$
(l) $y(t)=e^{-t}+2te^{-2t/3}$
(m) $x(t)=\frac{5}{4}+\frac{1}{2}t-e^{t}+\frac{5}{12}e^{2t}-\frac{2}{3}e^{-t}$
(n) $x(t)=\frac{9}{20}e^{-t}-\frac{7}{16}\cos t+\frac{25}{16}\sin t-\frac{1}{80}\cos 3t$
$-\frac{3}{80}\sin 3t$

6 (a) $x(t)=\frac{1}{4}(\frac{15}{4}e^{3t}-\frac{11}{4}e^{t}-e^{-2t})$, $y(t)=\frac{1}{8}(3e^{3t}-e^{t})$
(b) $x(t)=5\sin t+5\cos t-e^{t}-e^{2t}-3$
$y(t)=2e^{t}-5\sin t+e^{2t}-3$
(c) $x(t)=3\sin t-2\cos t+e^{-2t}$
$y(t)=-\frac{7}{2}\sin t+\frac{9}{2}\cos t-\frac{1}{2}e^{-3t}$
(d) $x(t)=\frac{3}{2}e^{t/3}-\frac{1}{2}e^{t}$, $y(t)=-1+\frac{1}{2}e^{t}+\frac{3}{2}e^{t/3}$
(e) $x(t)=2e^{t}+\sin t-2\cos t$
$y(t)=\cos t-2\sin t-2e^{t}$
(f) $x(t)=-3+e^{t}+3e^{-t/3}$
$y(t)=t-1-\frac{1}{2}e^{t}+\frac{3}{2}e^{-t/3}$
(g) $x(t)=2t-e^{t}+e^{-2t}$, $y(t)=t-\frac{7}{2}+3e^{t}+\frac{1}{2}e^{-2t}$
(h) $x(t)=3\cos t+\cos(\sqrt{3}t)$
$y(t)=3\cos t-\cos(\sqrt{3}t)$
(i) $x(t)=\cos(\sqrt{\tfrac{3}{10}}t)+\frac{3}{4}\cos(\sqrt{6}t)$
$y(t)=\frac{5}{4}\cos(\sqrt{\tfrac{3}{10}}t)-\frac{1}{4}\cos(\sqrt{6}t)$
(j) $x(t)=\frac{1}{3}e^{t}+\frac{2}{3}\cos 2t+\frac{1}{3}\sin 2t$
$y(t)=\frac{2}{3}e^{t}-\frac{2}{3}\cos 2t-\frac{1}{3}\sin 2t$

7 $I_1(s)=\dfrac{E_1(50+s)s}{(s^2+10^4)(s+100)^2}$

$I_2(s)=\dfrac{Es^2}{(s^2+10^4)(s+100)^2}$

$i_2(t)=E(-\frac{1}{200}e^{-100t}+\frac{1}{2}te^{-100t}+\frac{1}{200}\cos 100t)$

9 $i_1(t)=20\sqrt{\tfrac{1}{7}}e^{-t/2}\sin(\frac{1}{2}\sqrt{7}t)$

10 $x_1(t)=-\frac{-3}{2}\cos(\sqrt{3}t)-\frac{-7}{10}\cos(\sqrt{13}t)$
$x_2(t)=-\frac{1}{2}\cos(\sqrt{3}t)+\frac{3}{2}\cos(\sqrt{13}t)$, $\sqrt{3}$, $\sqrt{13}$

11.5 Review exercises

1 (a) $x(t)=\cos t+\sin t-e^{-2t}(\cos t+3\sin t)$
(b) $x(t)=-3+\frac{13}{7}e^{t}+\frac{15}{7}e^{-2t/5}$

2 (a) $e^{-t}-\frac{1}{2}e^{-2t}-\frac{1}{2}e^{-t}(\cos t+\sin t)$
(b) $i(t)=4e^{-t}-3e^{-2t}$
$+V[e^{-t}-\frac{1}{2}e^{-2t}-\frac{1}{2}e^{-t}(\cos t+\sin t)]$

3 $x(t)=-t+5\sin t-2\sin 2t$,
$y(t)=1-2\cos t+\cos 2t$

4 $\frac{1}{5}(\cos t+2\sin t)$
$e^{-t}[(x_0-\frac{1}{5})\cos t+(x_1+x_0-\frac{3}{5})\sin t]$
$\sqrt{\tfrac{1}{5}}$, 63.4° lag